## Math 40960, Topics in Geometry

## Problem Set 1, due February 19, 2014

Note: Answer the questions as I've written them here; the references to the books are just so you know where they came from. Of course you should show all your work. And as always, if you get any help from any source, in person or online or otherwise, you need to acknowledge it.

There are currently 7 problems.

1. (Hartshorne p. 25, Exercise 2.15) Suppose your straightedge is only about two inches long, but you have a fully working compass. Suppose also that on your sheet of paper you have two points that are about 3 inches apart. Carefully explain how you can use this "short straightedge" and compass, to construct the line segment joining the two points. Along the way, state and prove any lemmas you need to reach your final conclusion. Also, indicate if there is any "trial and error" that you need along the way.

Nota bene: The lengths of 2 inches and 3 inches stated in the problem are only approximations, and your answer should not at all rely on these measurements. But you can assume that the straightedge is more than half the distance between the two points, if you want.
2. (Stillwell p. 9, Exercise 1.3.5) In the text, the author shows how to divide a line segment into $n$ equal parts. In particular, when $n=3$, he shows how to trisect a given line segment. Now look at the following diagram (or Figure 1.10 in Stillwell). Suppose that you trisect the line segment joining $A$ and $B$, producing points $P$ and $Q$ that are $\frac{1}{3}$ and $\frac{2}{3}$ along the segment, respectively. Let $O$ be on the perpendicular bisector (not shown) of $A B$. Give a proof of the fact that the lines $O P$ and $O Q$ do not trisect the angle $A O B$. (I want more than just "from the picture it's clear that the angles are not equal." I want a proof. Suppose that the angles are equal, and derive a contradiction.)

3. Give an example of a partial linear space $(\mathcal{P}, \mathcal{L})$ whose dual is not a partial linear space. Include the incidence diagram for both $(\mathcal{P}, \mathcal{L})$ and its dual.
4. Consider the partial linear space defined by the following incidence diagram:


Notice that points are in capitals and lines are in lower case, and that there is no additional connection between, for example, $P$ and $p$ than is indicated in the diagram. Notice also that $\ell$ contains three points while all other lines contain two.
(a) Draw the incidence diagram for the dual of this partial linear space. In the dual diagram, please use capital letters for the lines and lower case for the points.
(b) Describe the automorphism group of this partial linear space as a subgroup of the permutation group $S_{5}$. In particular, how many elements does it have?
5. (Hartshorne p. 71, Exercise 6.1) Recall that an incidence geometry is a linear space (following Moorhouse) together with the axiom (I3) that there exist three non-collinear points.
(a) Describe all possible incidence geometries on a set of four points, up to isomorphism. This includes a proof that you have found all of them, and that no two on your list are isomorphic.
(b) Which ones in your answer to (a) also satisfy Playfair's axiom (what we called WP in class) (see Hartshorne p. 68)?
6. Recall:
(I1), (I2), (I3) $\Longleftrightarrow$ incidence geometry
(I1), (I2), (I3), (P) $\Longleftrightarrow$ (AP1), (AP2), (AP3) $\Longleftrightarrow$ affine plane
We showed in class that in an affine plane, parallelism is an equivalence relation.
(a) Give an example to show that in an incidence geometry, parallelism is not necessarily an equivalence relation.
(b) On the other hand, suppose that you have an incidence geometry in which parallelism is an equivalence relation. Prove that (WP) must hold.
(c) Give an example of an incidence geometry in which parallelism is an equivalence relation but (P) does not hold.
7. (Moorhouse Exercise 2.2) The following is an incidence diagram for the affine plane of order 3. Notice that

- there are nine points and twelve lines;
- each line contains exactly three points.
(This is not part of what you have to prove - we're just making an observation.)


In case it's hard to read, the middle point is $(1,1)$. The goal of this exercise is to show that the affine plane of order 3 embeds in the complex affine plane $\mathbb{A}^{2}(\mathbb{C})$. In particular, we require that the indicated lines are really lines in the complex plane, i.e. have equations of the form $a x+b y=c$ where $a, b$ and $c$ are complex numbers (possibly real).

To get started, we have given coordinates to five of the nine points. The goal is to give complex (possibly real) coordinates for the remaining four points. The following is a suggested way to get started, but what we're looking for is the answer to ( 7 d ).
Warning: the points you are looking for do not have entirely real coordinates. You don't need to know anything about complex variables beyond the quadratic formula when there are no real roots.
(a) Why aren't the missing points just $(1,2),(0,1),(1,0)$ and $(2,1)$ like the picture seems to suggest? (Again, beware of relying too much on a picture!)
(b) For six of the lines, it's easy to find the equation. Find these six lines.
(c) As a result, for each of the four missing points, you know one of the two coordinates almost for free. What are these?
(d) Find the missing coordinates and tell me what the four unlabeled points are. (This may take some work.)

